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Mean-first-passage times for systems driven by the coin-toss square wave

Josep M. Porrà and Jaume Masoliver
Departament de Física Fonamental, Universitat de Barcelona, Diagonal, 647, 08028-Barcelona, Spain

Katja Lindenberg
Department of Chemistry and Institute for Nonlinear Science, University of California, San Diego, La Jolla, California 92093-0340
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We study the mean-first-passage-time problem for systems driven by the coin-toss square-wave signal. Exact analytic solutions are obtained for the driftless case. We also obtain approximate solutions for the potential case. The mean-first-passage time exhibits discontinuities and a remarkable nonsmooth oscillatory behavior which, to our knowledge, has not been observed for other kinds of driving noise.

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I. INTRODUCTION

There is a vast literature dealing with dynamical systems driven by Gaussian white noise, indeed a venerable literature that extends back more than a century. In recent years interest has broadened to processes that are driven by noise that is not white and/or not Gaussian, a problem that has also generated a considerable literature [1]. Nonwhite noise (colored noise) represents random processes whose time scales are not appreciably shorter than those characteristic of the dynamical system of interest. Because Gaussian colored noise is very difficult to deal with analytically, a great deal of work has also been done with driving noise that is statistically much simpler than Gaussian noise, namely, dichotomous noise. Not only is it often simpler to deal with noise that can take on only two values (rather than an infinity or a continuum of values), but in many situations in which the important characteristic of the noise is whether it is "on" or "off," or whether it is "up" or "down," such two-valued noise is actually a more accurate representation [1–7].

A model used in many of these studies [2–4] is a single-degree-of-freedom system described by a dynamical variable $X(t)$ and driven by a noise $F(t)$:

$$\dot{X}(t) = f(X) + F(t). \tag{1.1}$$

The problem is completely specified once the function $f(X)$ and the statistics of the noise $F(t)$ are given. Further, one has to decide what particular property or characteristic of the "solution" $X(t)$, itself a random variable, is to be calculated. Common among these are the probability density $p(x,t)$ for the probability that the random variable $X(t) = x$, the stationary form $p(x)$ of this density as $t \to \infty$, and the mean-first-passage time (MFPT) for $X(t)$ to first achieve given threshold values. More recently, work has been done on systems such as (1.1) but with the inclusion of a double-time-derivative term in $X(t)$, that is, mechanical systems that include inertial effects [5,6]. Equation (1.1) can be thought of as a highly overdamped limit of such a mechanical system.

In this paper we consider the MFPT problem for stochastic systems governed by an evolution equation of the form (1.1) where $f(X)$ is a smooth function and $F(t)$ is a particular kind of dichotomous noise that has seldom been considered in the literature. The virtue of our noise, as we show below, is that in the appropriate parameter limits it encompasses a perfectly periodic signal, the noise appropriate to an ordinary random walk (that is, as generated by flipping a fair coin), and a standard dichotomous Markov process.

The noise that we consider can be called "persistent-periodic" dichotomous noise: $F(t)$ can take on either of the two values $F(t) = \pm a$ during *fixed time intervals* of duration τ. At the end of each time interval of duration τ there is a probability of persistence p that the noise retains the same value and a probability $q = 1 - p$ that the noise switches to the other value. In other words, the noise is symmetric and its transition-probability matrix is given by

$$\begin{pmatrix} p & q \\ q & p \end{pmatrix}. \tag{1.2}$$

Over 30 years ago the stochastic system defined by Eqs. (1.1) and (1.2) was first studied by McFadden [8] and Cohen [9] in the context of *RC* low-pass filters, i.e., with a linear drift. In that work the driving noise $F(t)$ was called "the coin-toss square wave." The main result of those analyses was the calculation of the stationary probability distribution function (PDF) of the process (1.1). McFadden only treated the case of the symmetric coin-toss square wave ($p = q = \frac{1}{2}$) and evaluated the stationary PDF for one specific value of the parameters. Cohen extended the results to the general case $p \neq q$ and for a wider parameter regime.

More recently Irwin, Fraser, and Kapral [4] and Fraser and Kapral [5] also treated a dichotomous driving noise and a linear drift, but with a different transition-probability matrix

$$\begin{pmatrix} p & q \\ p & q \end{pmatrix}. \tag{1.3}$$

That is, at each time interval one value is selected with probability p and the other with probability $q = 1 - p$, regardless of the previous value of the noise. This kind of noise might be called "periodic" dichotomous noise since there is no memory or persistence of the previous value. Like McFadden and Cohen, Fraser and co-workers studied certain features of the stationary probability-density function for the process (1.3).

The results found by Cohen [9] and by Fraser and co-workers [4,5] for these systems are dramatic. For example, we cite one result (found independently by both groups): for a linear drift and certain parameter values the set of points visited by the system after an infinite number of switches of the noise is a Cantor set. Fraser and co-workers observed a broader array of unexpected behaviors, including resonancelike phenomena that they termed "stochastically induced coherence."

It is useful at this point to briefly anticipate our own most interesting result: we find that the MFPT for our system can exhibit very unusual behavior even in the apparently simplest situations (e.g., in the absence of drift), behavior that is entirely different from that of the system (1.1) driven by dichotomous Markov noise [2] or by any continuous form of the noise (such as Gaussian noise). In particular, we calculate the MFPT to reach either of two boundaries if the process starts at $X(0) = x_0$. As a function of x_0 we find that the MFPT is an *oscillatory* function, that is, as the starting point of the process moves away from one boundary and closer to the other, the MFPT increases, then decreases, then increases, etc. Indeed, this oscillatory behavior is in general quite abrupt in the sense that a small change in the initial state can cause a considerable increase or decrease in the MFPT to reach a boundary.

The noise $F(t)$ defined by (1.2) reduces in various limits to several important and familiar cases. Thus, when $p = 0$, $F(t)$ becomes a periodic deterministic signal of period τ [note that the noise defined by (1.3) does not become strictly periodic in any limit]. When $p = q = 1/2$ (and only then), the noises defined by (1.2) and (1.3) agree. In this case when $f(X) \equiv 0$, the output $X(t)$ of Eq. (1.1) is the ordinary random walk. For $p \neq q$ the output $X(t)$ of (1.1) with (1.2) is the persistent random walk. Moreover, as we explain in Sec. III, if

$$p = 1 - \lambda\tau, \qquad q = \lambda\tau \qquad (\lambda > 0), \tag{1.4}$$

then in the limit $\tau \to 0$ the noise $F(t)$ defined by (1.2) becomes Markovian dichotomous noise.

The paper is organized as follows. In Sec. II we detail the dynamics of the system and derive the general equations satisfied by the MFPT. These equations turn out to be exact difference equations (as opposed to the usual differential equations associated with Gaussian white noise or with dichotomous Markov noise). In Sec. III we obtain closed analytical solutions of these equations when $f(X) = 0$ and study some relevant special cases. In Sec. IV we treat the small-τ case when $f(X) \neq 0$. In this case we recover the known results for dichotomous noise and Gaussian white noise. The conclusions are drawn in Sec. V.

II. ANALYSIS

We assume that $f(X)$ is a smooth function such that the solution $X(t)$ of Eq. (1.1) does not become infinite for a finite time. Let $X^+(t)$ and $X^-(t)$ be the solutions of Eq. (1.1) when $F(t) = a$ and $F(t) = -a$, respectively. We see from Eq. (1.1) that $X^{\pm}(t)$ is defined by the expression

$$t = \int_{x_0}^{X^{\pm}(t)} \frac{dx}{f(x) \pm a}, \tag{2.1}$$

where $x_0 = X(t = 0)$. Since $f(x) + a \geq f(x) - a$ for all x, we have by the *comparison theorem* [10] that

$$X^+(t) \geq X^-(t). \tag{2.2}$$

Let x_s be an asymptotically fixed stable point of Eq. (1.1), i.e., when $x_s = x_s^+$ then $f(x_s^+) + a = 0$ and

$$\lim_{t\to\infty} X^+(t) = x_s^+, \tag{2.3}$$

with similar relations for $F(t) = -a$ and $x_s = x_s^-$. Then by the comparison theorem we have

$$x_s^+ \geq x_s^-. \tag{2.4}$$

We note that this inequality necessarily implies that

$$f(x) + a \geq 0 \qquad \text{and} \qquad f(x) - a \leq 0 \tag{2.5}$$

for all x such that $x_s^- \leq x \leq x_s^+$. We thus see that when the process (1.1) has at least two fixed points (one for the value $+a$ and the other for $-a$) there exist two "natural boundaries," x_s^- and x_s^+, that the system cannot cross. Therefore if we are interested in finding the MFPT for the process to reach certain values, say z_2 or z_1, these values must lie inside the natural barriers, that is,

$$x_s^- < z_1 \leq z_2 < x_s^+. \tag{2.6}$$

If there are no fixed points we assume that $X^+(t)$ [$X^-(t)$] is an increasing (decreasing) function of time. In this case no restriction applies to the critical values z_2 and z_1. The various quantities defined in this paragraph are sketched in Fig. 1.

In order to establish the equations satisfied by the MFPT to z_2 or z_1 we first need to define some dynamical quantities. Let $\Delta x^{\pm}(x_0)$ be the distances traveled by the system when driven by $F(t) = \pm a$ during a period τ, i.e.,

$$\tau = \int_{x_0}^{x_0+\Delta x^{\pm}(x_0)} \frac{dx}{f(x) \pm a}. \tag{2.7}$$

Furthermore, let x^+ (x^-) represent the farthest that the process can be from the boundary z_2 (z_1) if it is to reach this boundary *within one period* τ:

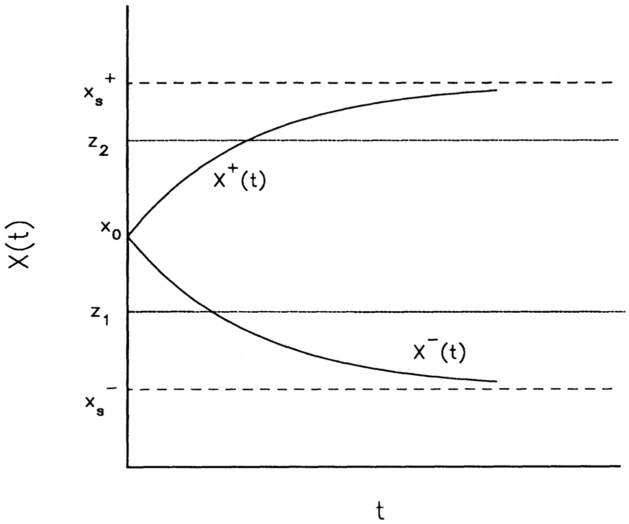

FIG. 1. Dynamical variable as a function of time for two values of $F(t)$. $X^+(t)$ is the trajectory when $F(t) = a$, and $X^-(t)$ is the trajectory when $F(t) = -a$. The trajectories approach their respective asymptotic fixed points x_s^+ and x_s^-. The critical values are z_1 and z_2.

$$\tau = \int_{x^+}^{z_2} \frac{dx}{f(x)+a}, \qquad \tau = \int_{x^-}^{z_1} \frac{dx}{f(x)-a}. \tag{2.8}$$

We now define $T^+(x_0)$ and $T^-(x_0)$ to be the MFPT's to either of the critical boundaries z_2 and z_1 under the assumption that $F(0) = +a$ and $F(0) = -a$, respectively. If we assume that $\text{Prob}\{F(0) = \pm a\} = \frac{1}{2}$ then the MFPT to either boundary averaged over the initial value of the driving noise is given by

$$T(x_0) = \frac{T^+(x_0) + T^-(x_0)}{2}. \tag{2.9}$$

We begin with the choice $F(0) = +a$. If $x_0 > x^+$ then the system (1.1) crosses the upper level z_2 during the first time interval with certainty (i.e., in a time smaller than τ). Therefore

$$T^+(x_0) = \int_{x_0}^{z_2} \frac{dx}{f(x)+a} \quad (x_0 > x^+). \tag{2.10}$$

On the other hand, if $x_0 \le x^+$ the system reaches z_2 during a later period. In this case we may write

$$\begin{aligned} T^+(x_0) = \tau + pT^+(x_0 + \Delta x^+(x_0)) \\ +qT^-(x_0 + \Delta x^+(x_0)), \end{aligned} \tag{2.11}$$

where $x_0 \le x^+$. This equation is derived from the consideration that if $x_0 \le x^+$ then the MFPT is necessarily greater than τ. Moreover, the second term on the right-hand side of Eq. (2.11) is the MFPT [starting from $x_0 + \Delta x^+(x_0)$ and with $F(t) = +a$] times the probability p of keeping the value $F(t) = +a$ for the driving noise. The third term on the right-hand side of Eq. (2.11) is the MFPT [starting from the same point, but with $F(t) = -a$] times the probability q of changing the value of the driving noise to $-a$.

Following analogous reasoning we find that

$$T^-(x_0) = \int_{x_0}^{z_1} \frac{dx}{f(x)-a} \quad (x_0 < x^-) \tag{2.12}$$

and

$$\begin{aligned} T^-(x_0) = \tau + pT^-(x_0 + \Delta x^-(x_0)) \\ +qT^+(x_0 + \Delta x^-(x_0)) \end{aligned} \tag{2.13}$$

when $x_0 \ge x^-$.

In the special case $p = q = \frac{1}{2}$ we can easily obtain a closed difference equation for the MFPT (2.9) to either boundary. In this case it is straightforward to find that Eqs. (2.11) and (2.13) are equivalent to

$$2T(x_0) - T(x_0 + \Delta x^+(x_0)) - T(x_0 + \Delta x^-(x_0)) = 2\tau, \tag{2.14}$$

when $x^- \le x_0 \le x^+$. Now the "end-interval conditions" (2.10) and (2.12) (which we will loosely call "boundary conditions") are

$$T(x_0) = \tfrac{1}{2} \int_{x_0}^{z_2} \frac{dx}{f(x)+a} + \tfrac{1}{2}\left[\tau + T(x_0 + \Delta x^-(x_0))\right], \tag{2.15}$$

when $x^+ < x_0 \le z_2$, and

$$T(x_0) = \tfrac{1}{2} \int_{x_0}^{z_1} \frac{dx}{f(x)-a} + \tfrac{1}{2}\left[\tau + T(x_0 + \Delta x^+(x_0))\right], \tag{2.16}$$

when $z_1 \le x_0 < x^-$. Equation (2.15) is derived from the consideration that if $x^+ < x_0 \le z_2$ then $T^-(x_0)$ is given by Eq. (2.13) while $T^+(x_0)$ is given by Eq. (2.10). Combining these two equations to obtain the MFPT (2.9) then immediately leads to Eq. (2.15). Analogous reasoning leads to Eq. (2.16).

The most general statement of the MFPT problem considered here is contained in the difference equations (2.11) and (2.13) with the "boundary conditions" (2.10) and (2.12), respectively. This is a system of difference equations with *variable delay*, that is, the distances $\Delta x^\pm$ between the points connected by the equations change as x_0 changes, a consequence of the presence of $f(x)$. This system of difference equations replaces the more traditional second-order differential equation that would be appropriate for the MFPT if the driving noise were Gaussian and white. Difference equations with variable delay are of course extremely difficult to solve analytically, so a solution in all generality is impossible. Indeed, an explicit solution for any nonzero form of $f(x)$ appears impossible—only a numerical solution appears feasible.

As we will see in Sec. III, for the driftless case $[f(x) = 0]$ the set of difference equations (2.10)–(2.13) with *variable delay* reduces to a system of difference equations with *constant delay*, that is, to a set of ordinary difference equations. In this case it is not difficult to find an exact solution because the set $\Omega(x_0)$ of points visited by the system after an infinite number of iterations of period τ (starting from a given "initial position" x_0) is a *finite* set. In the presence of any drift $f(x) \neq 0$, however, the situation becomes much more complex, since now the delay $\Delta^{\pm}(x_0)$ depends on the position and $\Omega(x_0)$ is in general an infinite set. As mentioned earlier and as an illustration of the complexity of the general case we mention that, for a linear drift, Cohen [9] found that for certain parameter values the set of points $\Omega_0 \equiv \{\Omega(x_0) | x_0 \in [\alpha, \beta]\}$ visited by the system after an infinite number of iterations of period τ, starting from a given interval (α, β), is a Cantor set that is known to have a fractal dimension $0 < d < 1$. The same result has been obtained independently by Fraser and co-workers [4,5].

III. LINEAR DRIFTLESS DYNAMICS

For the driftless case, $f(x) = 0$, the dynamics of the system is linear since in this case $X^{\pm}(t) = x_0 \pm at$ and $\Delta x^{\pm}(x_0) = \pm a\tau$. We can assume without loss of generality that $z_1 = 0$, $z_2 = L$, and $a = \tau = 1$. This last assumption is equivalent to the choice of the dimensionless quantities

$$x' = \frac{x}{a\tau}, \qquad t' = \frac{t}{\tau}. \tag{3.1}$$

Below we drop the primes.

Equation (2.11) now reads

$$T^{+}(x_0) = 1 + pT^{+}(x_0+1) + qT^{-}(x_0+1), \tag{3.2}$$

where $0 \leq x_0 \leq L-1$. In this case the boundary condition (2.10) is

$$T^{+}(x_0) = L - x_0 \qquad (L-1 < x_0 \leq L). \tag{3.3}$$

Also from Eqs. (2.12) and (2.13) we have

$$T^{-}(x_0) = 1 + pT^{-}(x_0-1) + qT^{+}(x_0-1) \tag{3.4}$$

when $1 \leq x_0 \leq L$, with

$$T^{-}(x_0) = x_0 \qquad (0 \leq x_0 < 1). \tag{3.5}$$

Equations (3.2) and (3.4) constitute a set of linear coupled difference equations. From this set together with the boundary conditions (3.3) and (3.5) we will now find closed equations for both $T^{+}(x_0)$ and $T^{-}(x_0)$ for all x_0. Toward this purpose, Eq. (3.2) can be rearranged and its arguments shifted to read

$$T^{-}(x_0) = \frac{1}{q}[T^{+}(x_0-1) - pT^{+}(x_0) - 1] \qquad (1 \leq x_0 \leq L) \tag{3.6}$$

and also

$$T^{-}(x_0-1) = \frac{1}{q}[T^{+}(x_0-2) - pT^{+}(x_0-1) - \tau] \qquad (2 \leq x_0 \leq L). \tag{3.7}$$

Substituting Eqs. (3.6) and (3.7) into Eq. (3.4) we obtain the following closed difference equation for $T^{+}(x_0)$:

$$2T^{+}(x_0-1) - T^{+}(x_0) - T^{+}(x_0-2) = 2\frac{q}{p}, \tag{3.8}$$

where $2 \leq x_0 \leq L$. The full solution of this system requires two boundary conditions. One boundary condition is given by Eq. (3.3). In order to find the second boundary condition we substitute Eq. (3.6) into Eq. (3.4) and obtain

$$\frac{1}{q}[T^{+}(x_0-1) - pT^{+}(x_0) - 1] = 1 + qT^{+}(x_0-1) + pT^{-}(x_0-1),$$

valid for $1 \leq x_0 \leq L$. If we restrict x_0 to the interval $1 \leq x_0 < 2$ then $0 \leq x_0 - 1 < 1$ and from Eq. (3.5) we have

$$T^{-}(x_0-1) = x_0 - 1.$$

The second boundary condition then reads

$$(1-q^2)T^{+}(x_0-1) - pT^{+}(x_0) = (1+q) + qp(x_0-1),$$

where $1 \leq x_0 < 2$, or equivalently,

$$(1-q^2)T^{+}(x_0) - pT^{+}(x_0+1) = (1+q) + qpx_0, \tag{3.9}$$

where $0 \leq x_0 < 1$.

The solution of the second-order difference equation (3.8) with the boundary conditions (3.3) and (3.9) is detailed in Appendix A. To exhibit the solution it is convenient to introduce the probability ratio

$$\xi = \frac{q}{p}. \tag{3.10}$$

A perfectly periodic signal then corresponds to the value $\xi = \infty$, an ordinary random walk to $\xi = 1$, and a totally persistent walk with a fixed velocity to $\xi = 0$. In terms of this ratio, the results for the MFPT to 0 or L for the driftless process then are

$$T^{+}(x_0) = L - (x_0 + j) + Aj - \xi j^2, \tag{3.11}$$

where

$$A = \begin{cases} 1 + N\xi + \dfrac{\xi}{1+N\xi}[2(x_0+j) - L - N + 1] & \text{if } L-(j+1) < x_0 < N-j \\ 1 + (N+1)\xi + \dfrac{\xi}{1+(N+1)\xi}[2(x_0+j) - L - N] & \text{if } N-j \leq x_0 \leq L-j. \end{cases} \tag{3.12}$$

Here $j \equiv [L - x_0]$ is the integer part of $L - x_0$ and is thus the number of points in the interval $(x_0, L]$ where the random walker may change its velocity. Depending on the initial position x_0, j can take on the integer values $j = 0, 1, 2, \ldots, N$ where N is the integer part of L. Similarly we find

$$T^-(x_0) = x_0 - k + Bk - \xi k^2, \tag{3.13}$$

where

$$B = \begin{cases} 1 + (N+1)\xi + \dfrac{\xi}{1+(N+1)\xi}\,[L - 2(x_0 - k) - N] & \text{if } k \le x_0 \le L - (N-k) \\ 1 + N\xi + \dfrac{\xi}{1+N\xi}\,[L - 2(x_0 - k) - (N-1)] & \text{if } L - (N-k) < x_0 < k+1. \end{cases} \tag{3.14}$$

Here $k \equiv [x_0]$ is the integer part of x_0 representing the number of points where the random walker may change its velocity in the interval $[0, x_0)$. We note that $j + k + 1$ *is the total number of points in the interval* $[0, L]$ *where the system may change its velocity.*

Note that the MFPT's are quadratic in the integer part of the distance to one or the other boundary. Quadratic dependence on the distance is a signature of a driftless process (such as the mean-square displacement being proportional to the time in an ordinary diffusive process), but the dependence on only the *integer part* here leads to very interesting and different behavior.

Let us analyze these results with the aid of Figs. 2–4. The most dramatic observation is the oscillatory behavior of the MFPT as a function of the initial position x_0 of the walker. This unusual behavior will be addressed in detail below, where a physical explanation of this observation is given. First we note some of the more obvious features of our results. We see in Fig. 2 that for a fixed L the MFPT for a given initial position is in general larger for smaller p (larger ξ). This is reasonable, since a perfectly periodic signal ($\xi = \infty$) causes the walker to simply move back and forth, repeatedly covering the same ground within the interval unless the walker steps on a boundary during the first period. Thus, unless the walker reaches the boundary during the first period it will never reach it at all and thus the MFPT T^+ or T^- is then infinite. The other extreme, $\xi = 0$, leads to a walk directly toward a boundary with the initial velocity. Note also the effect of increasing the number of points at which the walker can change its velocity, as shown in Figs. 3 and 4. The MFPT is now greater since the walker can change direction more frequently and can therefore delay its arrival at the boundaries.

Let us return to the oscillatory behavior of the MFPT. This behavior arises from the discontinuities that can be

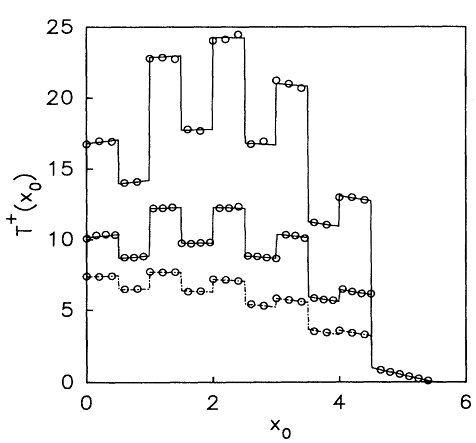

FIG. 2. Mean-first-passage time $T^+(x_0)$ in dimensionless units as a function of x_0 for $L = 5.5$. The number of points at which the walker can change its velocity between the absorbing boundaries is $N = 5$. Results are shown for three different values of the persistence ratio: $\xi = 7/3$ (top curve), 1 (middle curve), and 3/7 (bottom curve). The circles represent the results of direct numerical simulations of the process.

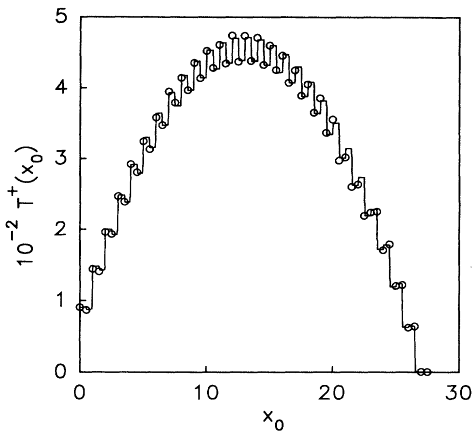

FIG. 3. Mean-first-passage time $T^+(x_0)$ in dimensionless units as a function of x_0 for $\xi = 7/3$ and $L = 27.5$. The number of points at which the walker can change its velocity between the absorbing boundaries is $N = 27$. The circles represent the results of direct numerical simulations.

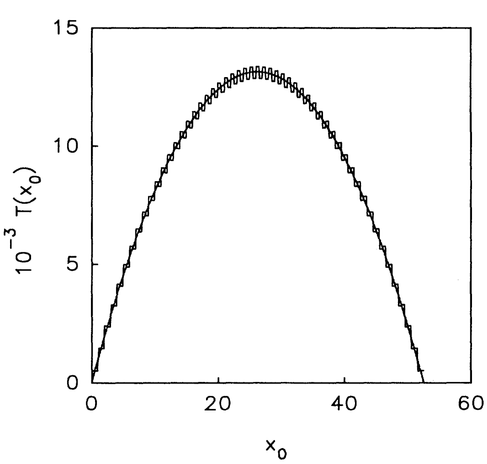

FIG. 4. Mean first-passage time $T(x_0)$ in dimensionless units as a function of x_0 for large ξ and $L = 52.5$ ($N = 52$). Here $p = 0.05$ ($\xi = 19$). The jagged curve is the exact result obtained from Eqs. (3.11)–(3.14) in Eq. (2.9) and the smooth curve is the approximation given by Eq. (4.29).

seen in Eq. (3.11) with (3.12) and Eq. (3.13) with (3.14) precisely because of the dependence on only the integer portion of the distance to one or the other boundary: both $T^+(x_0)$ and $T^-(x_0)$ are discontinuous from the right at $x_0 = L - N, L - (N-1), \ldots, L-1$ (where T^+ and T^- jump down) and discontinuous from the left at $x_0 = 1, \ldots, (N-1), N$ (where T^+ and T^- jump up). These discontinuities are evaluated explicitly in Appendix B.

Since the results obtained here and in Appendix B are analytically somewhat cumbersome, it is useful to present the simplified expression for the MFPT when the persistence probability p is small (ξ is large). For the initial-condition-averaged MFPT (2.9) we obtain in this limit

$$T(x_0) \sim \begin{cases} \frac{N-1}{2}(1+\xi) + \xi k(N-1-k) \\ \text{if } L-(N-k+1) < x_0 < k \\ \frac{N}{2}(1+\xi) + \xi k(N-k) \\ \text{if } k \le x_0 \le L-(N-k). \end{cases} \tag{3.15}$$

This equation is illustrated in Fig. 5. Note that although the absolute magnitude of the discontinuities in the MFPT increases with increasing ξ, the magnitude relative to the MFPT decreases.

Let us now return to the physical explanation of the oscillations apparent in Figs. 2–4. Let us first examine the behavior of $T^+(x_0)$ near a left discontinuity point, for example, $x_0 = 2$. When x_0 approaches 2 from the left we have that $k = 1$. However, when x_0 approaches 2 from the right we have $k = 2$. As a result of this, the random walker can change its velocity at one additional point, which implies that the escape time can increase by at least 2, which is the time to go away from and return to the same point. The net result of this effect is a positive discontinuity of $T^+(x_0)$ at $x_0 = n$ with $n = 1, 2, 3, \ldots, N$. On the other hand, if, for example, x_0 approaches $L-2$ from the left we have that $j = 2$, while approaching $L-2$ from the right we have $j = 1$. In this case the time to escape decreases by at least 2. This results in a negative discontinuity at $x_0 = L - n$ with $n = 1, 2, 3, \ldots, N$. The above arguments also apply to $T^-(x_0)$.

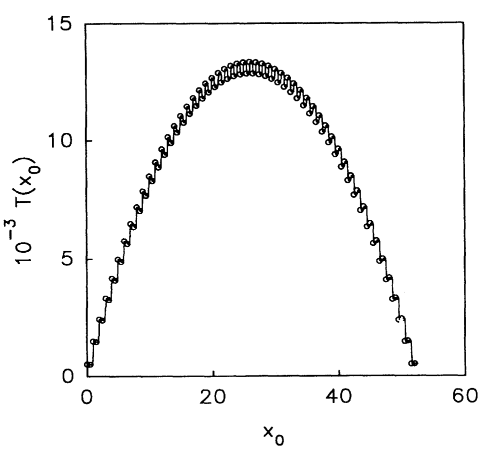

FIG. 5. Same as in Fig. 4 but now the circles are obtained from the approximation (3.15). In both cases the agreement is excellent.

We should note that the above discussion is only valid if L *is not an integer.* If L is an integer, i.e., if $L = N$, then a most interesting feature in the behavior of the MFPT is observed, namely, a collapse of pairs of discontinuities to single points. The MFPT is still given by Eqs. (3.11) and (3.13) but now the coefficients A and B are given by

$$A = \begin{cases} 1 + N\xi + \frac{\xi}{1+N\xi}[(2j+1) + 2x_0 - 2N] \\ \qquad \text{if } N-(j+1) < x_0 < N-j \\ 1 + (N+1)\xi \quad \text{if } x_0 = N-j. \end{cases}$$

$$B = \begin{cases} 1 + (N+1)\xi \quad \text{if } x_0 = k \\ 1 + N\xi + \frac{\xi}{1+N\xi}[(2k+1) - 2x_0] \\ \qquad \text{if } k < x_0 < k+1. \end{cases} \tag{3.16}$$

In this case the MFPT is discontinuous from the right and *also* from the left at the *same* points $x_0 = 1, 2, 3, \ldots, N$ as shown in Fig. 6. At these singular points the MFPT jumps up and then down again, so that the solid line *together with the dots* constitute the exact MFPT. Note also that the dots correspond to the MFPT for a random walk on a lattice with nonabsorbing lattice sites at positions $0, 1, \ldots, N$ and traps at -1 and $N+1$ (cf. below).

For completeness we also give the expression for the averaged MFPT (2.9) [cf. Eqs. (3.11) and (3.13)]:

$$T(x_0) = \tfrac{1}{2}\left[L - j - k + Aj + Bk - \xi\left(j^2 + k^2\right)\right], \tag{3.17}$$

with the values of A and B given in (3.16).

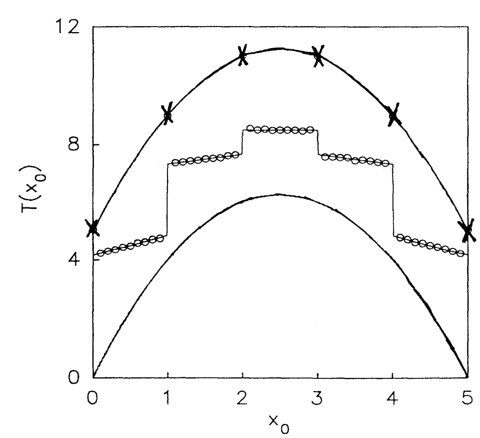

FIG. 6. Mean first-passage time $T^+(x_0)$ in dimensionless units as a function of x_0 when L is an integer. Here $L = N = 5$. The *middle* curve *together* with the $\times$'s are our exact results; the circles are direct simulations (note that $\times$'s are circled). The $\times$'s alone are also the MFPT's for an ordinary random walk as described in the text. The lower curve is the MFPT for a diffusion process with diffusion coefficient $D = 0.5$ and the upper curve is the aproximation given by Eq. (4.29).

A. Special cases

It is interesting and insightful to connect the general results obtained above with familiar ones obtained by other methods for special cases. These include the persistent random walk on a lattice, an ordinary random walk, and the continuous-time persistent random walk.

1. The persistent random walk on a lattice

Consider the case in which both L and x_0 are integers, i.e., $L = N$ and $x_0 \equiv m = 0, 1, 2, \ldots, N$. In this case Eq. (3.17) reads

$$T(m) = \frac{N}{2}(1+\xi) + \xi m(N-m), \tag{3.18}$$

which is precisely the MFPT for the persistent random walk on a lattice (the word persistent being associated with $p \neq \frac{1}{2}$, that is, with $\xi \neq 1$) [11].

2. The ordinary random walk

When $p = q = \frac{1}{2}$ one should recover from Eq. (3.18) the well-known expression for the MFPT for the ordinary random walk with traps at 0 and N [12]:

$$T_{0,N}^{\mathrm{RW}}(m) = m(N-m) \qquad (m = 0, 1, 2, \ldots, N). \tag{3.19}$$

However, when we set $p = q = \frac{1}{2}$ in Eq. (3.18) we obtain a different result:

$$T(m) = N + m(N-m) \qquad (m = 0, 1, 2, \cdots, N). \tag{3.20}$$

The reason for this difference is the following. In the derivation of Eq. (3.17) we have assumed that traps are located at $\lim_{\epsilon\to 0}(0-\epsilon)$ and $\lim_{\epsilon\to 0}(N+\epsilon)$ where $\epsilon > 0$, i.e., the traps are located immediately before 0 and immediately after N. Under this assumption Eq. (3.20) yields

$$T(0) \neq 0 \qquad \text{and} \qquad T(N) \neq 0, \tag{3.21}$$

while Eq. (3.19) necessarily implies that

$$T(0) = T(N) = 0. \tag{3.22}$$

For the ordinary random walk, the assumption that the traps are immediately before 0 and after N is equivalent to the assumption that *traps are located at* -1 *and* $N+1$. In this case T^{RW} is given by the MFPT to reach these traps *minus* the time expended in making one step to go from 0 to -1 or from N to $N+1$, that is,

$$T(m) = T_{-1,N+1}^{\mathrm{RW}}(m) - 1. \tag{3.23}$$

If we substitute Eq. (3.19) into the right-hand side of Eq. (3.23) we obtain Eq. (3.20) and our model reproduces the ordinary-random-walk result. This is the result shown by the dots in Fig. 6.

3. Continuous-time persistent random walk

The continuous-time persistent random walk was considered earlier by some of us [13]. It is equivalent to a random process $X(t)$ whose time evolution is governed by the dynamical equation

$$\dot{X}(t) = F(t),$$

where $F(t)$ is a Markovian dichotomous noise, i.e., a two-state noise with an exponential switching distribution [13–16]

$$\psi(t) = \lambda e^{-\lambda t}. \tag{3.24}$$

To establish the correspondence between the process considered in this paper and the continuous-time persistent random walk we take the continuum limit

$$N \to \infty, \qquad \tau \to 0, \qquad Na\tau \to L \quad \text{(finite)}. \tag{3.25}$$

We also rescale the probabilities p and q as [17]

$$p = 1 - \lambda\tau, \qquad q = \lambda\tau, \tag{3.26}$$

where $\lambda > 0$ is fixed. As a consequence of this scaling we observe that in the continuous limit (3.25), which we can also write in the form

$$\tau \to 0, \;\; m \to \infty, \;\; \text{with} \;\; m\tau = t \quad \text{(finite)}, \tag{3.27}$$

the probability $p^m = (1-\lambda\tau)^m$ that the noise keeps the value a during m steps goes to the distribution function of the Markovian dichotomous noise,

$$p^m \to e^{-\lambda t}, \tag{3.28}$$

which is precisely the cumulative distribution function associated with the switching distribution (3.24).

Substitution of (3.25) and (3.26) into (3.17) after reintroducing dimensioned quantities [cf. Eq. (3.1)] leads to the result

$$T(x_0) = \frac{L}{2a} + \frac{\lambda}{a^2} x_0 (L - x_0), \tag{3.29}$$

which agrees with previous results [15,16].

IV. DYNAMICS WITH ARBITRARY DRIFT IN THE HIGH-FREQUENCY LIMIT

In this section we consider the system (1.1) with a general drift $f(x)$. Without further approximation the analysis is necessarily numerical, and therefore we will mainly deal with the problem in the "high-frequency" (i.e., small-τ) limit. First, however, we wish to stress that in the presence of drift the system may again exhibit interesting behavior (such as that found by Cohen [9] and by Fraser and co-workers [4,5]), which shows up in unexpected ways in the MFPT. Thus, consider the case of a linear drift, $f(x) = -0.5x$. For the particular parameter values indicated in the figure caption, direct simulations lead to the data for the MFPT shown in Fig. 7. A noteworthy result is that the MFPT exhibits (in this case, two visible) sharp discontinuities, indicated by the dotted lines in the figure. The value of these discontinuities is in agreement with the theoretical values obtained from Eqs. (2.14)–(2.16). Thus, for instance, let us calculate the theoretical value of the discontinuity:

$$\Delta T(x^-) \equiv \lim_{\epsilon \to 0+} \left[T(x^- + \epsilon) - T(x^- - \epsilon)\right] . \quad (4.1)$$

From Eq. (2.14) we find that

$$\lim_{\epsilon\to 0+} T(x^- + \epsilon) = \tau + \tfrac{1}{2}[\lim_{\epsilon \to 0+} T(x^- + \Delta x^+(x^-) + \epsilon) + T(z_1)], \quad (4.2)$$

and from Eq. (2.15)

$$\lim_{\epsilon\to 0+} T(x^- - \epsilon) = \tau + \tfrac{1}{2} \lim_{\epsilon\to 0+} T(x^- + \Delta x^+(x^-) - \epsilon). \quad (4.3)$$

Hence

$$\Delta T(x^-) = \tfrac{1}{2}\left[T(z_1) + \Delta T(x^- + \Delta x^+(x^-))\right] , \quad (4.4)$$

where $\Delta T(x^- + \Delta x^+(x^-))$ is defined as in Eq. (4.1). The value of the discontinuities calculated numerically from Eq. (4.4) are in agreement with those from simulations. We note also that the magnitude of the discontinuity decreases as $T(z_1)$ decreases.

A second noteworthy feature in the figure is the possible evidence of oscillatory behavior. This oscillatory behavior, as well as the discontinuities discussed above, may be "remnants" of the behavior observed in the absence of a drift.

In order to carry out some analytic calculations, we must resort to approximations, here specifically for the small-τ limit. Our approach is based on expansions of the MFPT difference equations and boundary conditions in powers of τ, a procedure that is problematic in view of the discontinuities in the MFPT that we have just discussed. Such discontinuities would of course be entirely missed by a small-τ expansion. Indeed, in the presence of a drift, since the distance covered in a period τ depends on the position, the location of the discontinuities in each realization of the process that begins at x_0 and ends at one of the boundaries is different (this is not the case in the driftless problem). On the other hand, the relative magnitude of each discontinuity decreases and the number of discontinuities increases as the frequency τ^{-1} increases. In calculating the MFPT as an average of all these realizations it is not clear whether one ends up with a curve that is everywhere discontinuous or one that is smoothed out. We will see that the smoothing will become more effective as the frequency increases, so that the extreme smoothing implicit in the expansion procedure hopefully yields a good approximation to the actual MFPT curve. We will in any case check our approximate results against the exact ones found for the driftless problem and also against numerical simulations.

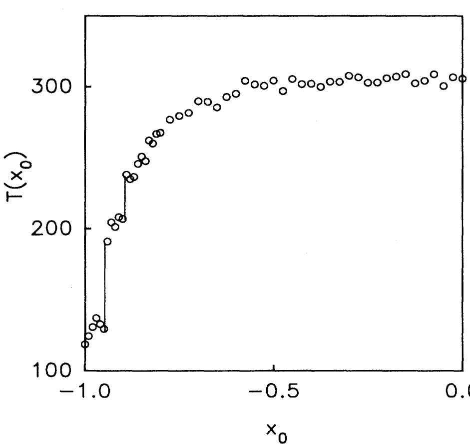

FIG. 7. Mean first-passage time $T(x_0)$ to $z_2 = -z_1 = 1$ for system (1.1) with linear drift $[f(x) = -0.5x]$ as a function of x_0 with $a = 1$, $\tau = 0.1$, and $p = 0.5$. The solid lines show the discontinuities of $T(x_0)$ at $x_0 = x^- = -0.949$ and $x_0 = -0.895$. The value of the discontinuities is related to the value of $T(z_1)$ [cf. Eq. (4.4)].

The starting point of our approximation schemes is the expansion of Eq. (2.7) in powers of τ:

$$\Delta x^{\pm}(x_0) = \tau[f(x_0) \pm a][1 + \tfrac{1}{2}\tau f'(x_0)] + O(\tau^3). \quad (4.5)$$

Before proceeding further we first note that the substitution of (4.5) into $T^{\pm}(x_0 + \Delta x^{\pm}(x_0))$ and subsequent expansion in powers of τ results in differential equations with small coefficients multiplying the highest derivative. These equations are known to be singular, with ordinary perturbation techniques yielding unsatisfactory results.

We have found only two special approximations that avoid the singularity when $\tau = 0$. One such approximation consists in expanding $T^{\pm}(x_0 + \Delta x^{\pm}(x_0))$ to *first order in* τ along with *a specific scaling of the probabilities p and q*. We will see next that in the limit $\tau \to 0$ this case reduces to that of dichotomous driving noise. A second special approximation consists of expanding $T^{\pm}(x_0 + \Delta x^{\pm}(x_0))$ to *second order in* τ but in order to avoid singularities *we also allow that* $a \to \infty$ *in such a way that the coefficient* $a^2\tau$ *multiplying the second derivative be finite*. We will see that when $\tau = 0$ this case reduces to the diffusion process (i.e., Gaussian white driving noise).

A. Dichotomous driving noise

Using Eq. (4.5) to first order in τ we find

$$T^+(x_0 + \Delta x^{\pm}(x_0)) = T^+(x_0) + \tau[f(x_0) \pm a]\frac{dT^+(x_0)}{dx_0} + O(\tau^2), \quad (4.6)$$

$$T^-(x_0 + \Delta x^{\pm}(x_0)) = T^-(x_0) + \tau[f(x_0) \pm a]\frac{dT^-(x_0)}{dx_0} + O(\tau^2).$$

Substituting this expansion into each of the terms in Eqs. (2.11) and (2.13) yields

$$\begin{aligned} T^{\pm}(x_0) &= \tau + pT^{\pm}(x_0) + qT^{\mp}(x_0) \\ &\quad + \tau[f(x) \pm a]\frac{d}{dx_0}[pT^{\pm}(x_0) + qT^{\mp}(x_0)] \\ &\quad + O(\tau^2). \end{aligned} \tag{4.7}$$

With the rescaling (3.26) in (4.7) and upon taking the limit $\tau \to 0$, we obtain

$$\frac{dT^{\pm}(x_0)}{dx_0} + \left[\frac{f'(x_0)}{f(x_0) \pm a} - 2\lambda \frac{f(x_0)}{f^2(x_0) - a^2}\right] \frac{dT^{\pm}(x_0)}{dx_0} = \frac{2\lambda}{f^2(x_0) - a^2}. \tag{4.8}$$

One set of boundary conditions associated with Eq. (4.6) immediately follows from Eqs. (2.10) and (2.12), namely,

$$T^{+}(z_2) = T^{-}(z_1) = 0. \tag{4.9}$$

The other set of boundary conditions associated with Eq. (4.8) is easily derived from Eqs. (2.11)–(2.13) and Eq. (4.9). Thus, substituting Eq. (4.6) into Eq. (2.11) and Eq. (2.13), using Eq. (4.9) and the scaling (3.26) we finally obtain

$$\left.\frac{dT^{+}(x)}{dx}\right|_{x=z_1} = \frac{1}{f(z_1) + a}[\lambda T^{+}(z_1) - 1] \tag{4.10}$$

and

$$\left.\frac{dT^{-}(x)}{dx}\right|_{x=z_2} = \frac{1}{f(z_2) - a}[\lambda T^{-}(z_2) - 1]. \tag{4.11}$$

These results are exactly those found in our earlier work by different methods [18].

B. Diffusionlike approximation

We next assume that τ is small and the magnitude a of the noise is large but now in such a way that $a\tau$ is small while

$$D \equiv \tfrac{1}{2}a^2\tau \tag{4.12}$$

is finite. Thus, starting from Eq. (4.5) up to first order in τ we have

$$\begin{aligned} T^{+}(x_0 + \Delta x^{\pm}(x_0)) &= T^{+}(x_0 \pm a\tau + \tau f(x_0)) + O(\tau^2), \\ T^{-}(x_0 + \Delta x^{\pm}(x_0)) &= T^{-}(x_0 \pm a\tau + \tau f(x_0)) + O(\tau^2), \end{aligned} \tag{4.13}$$

that is

$$\begin{aligned} T^{+}(x_0 + \Delta x^{\pm}(x_0)) &= T^{+}(x_0) + \tau\,[f(x_0) \pm a]\,\frac{dT^{+}(x_0)}{dx_0} \\ &\quad + \tfrac{1}{2}a^2\tau^2\frac{d^2T^{+}(x_0)}{dx_0} + O(\tau^2), \end{aligned} \tag{4.14}$$

$$\begin{aligned} T^{-}(x_0 + \Delta x^{\pm}(x_0)) &= T^{-}(x_0) + \tau\,[f(x_0) \pm a]\,\frac{dT^{-}(x_0)}{dx_0} \\ &\quad + \tfrac{1}{2}a^2\tau^2\frac{d^2T^{-}(x_0)}{dx_0} + O(\tau^2). \end{aligned} \tag{4.15}$$

Observe that under the above assumptions the term

$$\tfrac{1}{2}a^2\tau^2 = D\tau \tag{4.16}$$

is of first order in τ. Substituting Eq. (4.15) into each of the terms in Eqs. (2.11) and (2.13) we obtain

$$\begin{aligned} T^{\pm}(x_0) &= \tau + pT^{\pm}(x_0) + qT^{\mp}(x_0) \\ &\quad + \tau[f(x) \pm a]\frac{d}{dx_0}[pT^{\pm}(x_0) + qT^{\mp}(x_0)] \\ &\quad + D\tau\frac{d^2}{dx_0^2}[pT^{\pm}(x_0) + qT^{\mp}(x_0)] + O(\tau^2). \end{aligned} \tag{4.17}$$

Thus, in this special limit we have replaced the coupled difference equations (2.11) and (2.13) by two coupled second-order differential equations.

For the particular cases that we deal with subsequently it turns out to be more convenient to work instead with the initial-condition-averaged MFPT $T(x_0)$ defined in Eq. (2.9) and the difference (which will not enter in our subsequent specific cases)

$$S(x_0) \equiv \frac{T^{+}(x_0) - T^{-}(x_0)}{2}. \tag{4.18}$$

From Eq. (4.7) we see that these quantities obey the following set of second-order coupled differential equations (we only retain terms up to first order in τ):

$$D\frac{d^2T(x_0)}{dx_0^2} + f(x_0)\frac{dT(x_0)}{dx_0} + a(p - q)\frac{dS(x_0)}{dx_0} = -1 + O(\tau), \tag{4.19}$$

$$2qS(x_0) - a\tau\frac{dT(x_0)}{dx_0} = 0 + O(\tau^2). \tag{4.20}$$

From Eq. (4.20) we see that

$$\frac{dS(x_0)}{dx_0} = \frac{1}{2q}a\tau\frac{d^2T(x_0)}{dx_0^2}. \tag{4.21}$$

Substituting this equation into (4.19) we find

$$\frac{p}{q}D\frac{d^2T(x_0)}{dx_0^2} + f(x_0)\frac{dT(x_0)}{dx_0} = -1 + O(\tau). \tag{4.22}$$

This is precisely the form of the equation satisfied by the corresponding mean-first-passage time when $F(t)$ in the dynamical equation (1.1) is Gaussian white noise, but here the diffusion coefficient has been "rescaled" to

$$D_r \equiv \frac{p}{q}D = \frac{D}{\xi}, \tag{4.23}$$

where the parameter $\xi \equiv q/p$ was introduced in Sec. III.

Our next step is to find the boundary conditions that accompany Eq. (4.22). To this end we write Eq. (4.17) for $T^+(x_0)$ at $x_0 = z_1$ but only retain terms of order $a\tau$ (recall that $a\tau$ is much greater than τ):

$$T^+(z_1) = pT^+(z_1) + qT^-(z_1) + a\tau \frac{d}{dx_0}\left[pT^+(x_0) + qT^-(x_0)\right]\bigg|_{x_0=z_1} + O(\tau). \tag{4.24}$$

Now using Eq. (2.12) we see that $T^-(z_1) = 0$, $T^+(z_1) = 2T(x_0)$, and

$$\frac{dT^-(x_0)}{dx_0}\bigg|_{x_0=z_1} = -\frac{1}{f(x_0) - a} \simeq \frac{1}{a}, \tag{4.25}$$

where we have taken into account that $a \gg |f(x_0)|$ for all $x_0 \in [z_1, z_2]$. Hence

$$T(z_1) = \frac{p}{q} a\tau \frac{dT(x_0)}{dx_0}\bigg|_{x_0=z_1} + O(\tau). \tag{4.26}$$

Analogously we also find that

$$T(z_2) = -\frac{p}{q} a\tau \frac{dT(x_0)}{dx_0}\bigg|_{x_0=z_2} + O(\tau). \tag{4.27}$$

We finally observe that when $\tau \equiv 0$ we have

$$T(z_2) = T(z_1) = 0, \tag{4.28}$$

which are the boundary conditions for Gaussian white driving noise. The problem here is thus not simply equivalent to a rescaled version of the white-noise problem.

The solution of the system (4.22)–(4.27) is given in Appendix C. For the special (but common) case of an odd drift, $f(-x) = -f(x)$ and with the critical boundaries at $z_2 = z$ and $z_1 = -z$ the result simplifies to the rescaled form of the white-noise problem,

$$T(x_0) = \frac{1}{D_r}\int_{-z}^{x_0} dx\, e^{V(x)/D_r} \int_x^0 dx' e^{-V(x')/D_r} + \frac{p}{qD_r} a\tau\, e^{V(-z)/D_r} \int_{-z}^{0} dx\, e^{-V(x)/D_r} + O(\tau), \tag{4.29}$$

where $V(x)$ is the potential defined by

$$V(x) \equiv -\int^x dx' f(x'), \tag{4.30}$$

and D_r is given by Eq. (4.23).

In Figs. 8 and 9 we compare the results of direct simulations with the approximate form (4.29) for a linear drift. The triangles in Fig. 8 correspond to the simulation results with the parameters $a = 1$, $\tau = 0.1$, and $p = 0.5$ (the discontinuities apparent in Fig. 7 are clearly visible in the figure but have not been explicitly highlighted). Note that in this case $D = a^2\tau/2 = 0.05$ is not larger than $a\tau = 0.1$ and therefore (4.29) is not expected to provide a good approximation. Indeed it does not, as shown by the fact that the solid curve is quite far from the triangles. The circles in the figure are the results of simulations for the parameters $a = 3.162\,28$, $\tau = 0.01$, and $p = 0.5$. Now $D = 0.05$ is somewhat larger than $a\tau = 0.031\,622\,8$ (albeit not by much), and (4.29) is expected to be a better approximation. It is, as shown by the fact that the dashed line is closer to the simulations. The figure also shows the mean first-passage time for Gaussian white noise, which is the curve that the

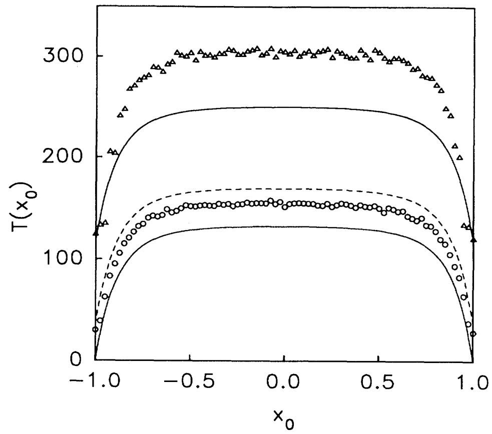

FIG. 8. Mean first-passage time $T(x_0)$ to $z_2 = -z_1 = 1$ for system (1.1) with linear drift $[f(x) = -0.5x]$ as a function of x_0. Two sets of parameters for the noise are used: (i) $a = 1$, $\tau = 0.1$, $p = 0.5$, and (ii) $a = 3.162\,28$, $\tau = 0.01$, $p = 0.5$. In both cases $D = 0.05$. Direct simulation (triangles for the first case and circles for the second one) as well as approximation (4.29) (upper solid curve, first case; dashed curve, second case) are plotted. The lower solid curve shows the result when $F(t)$ is Gaussian white noise.

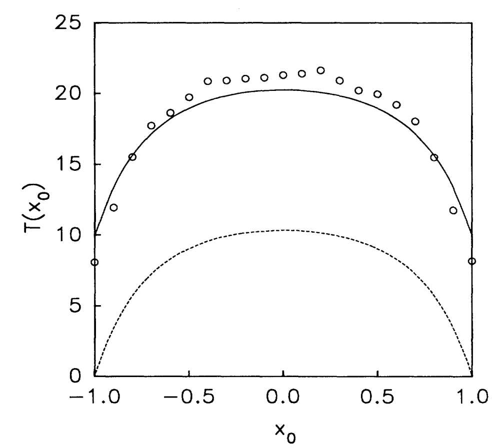

FIG. 9. Mean first-passage time $T(x_0)$ to $z_2 = -z_1 = 1$ for a system (1.1) with linear drift $[f(x) = -0.5x]$ as a function of x_0 with $a = 1$, $\tau = 0.1$, $p = 0.7$. The rescaled diffusion coefficient is $D_r = 0.35/3$. The circles are the simulation points while the solid line is the approximation given by Eq. (4.29). The dotted line has the same meaning as in Fig.7.

approximation (4.29) approaches with decreasing τ and increasing a (again, in the figure $D = 0.05$), that is, the diffusive MFPT

$$T(x_0) = \frac{1}{D_r} \int_{-z}^{x_0} dx\, e^{V(x)/D_r} \int_{x}^{0} dx' e^{-V(x')/D_r}. \qquad (4.31)$$

When $p \neq q$ (Fig. 9) there is an increase or decrease of the effective diffusion coefficient as given in Eq. (4.12). Thus if $p > q$ we have $D_r > D$, while if $p < q$ then $D_r < D$. The consequences of a change in the effective diffusion coefficient are exponentially large in the MFPT, as can be seen clearly by comparing the scales in Figs. 7 and 9. The parameters in the two figures are the same except for the value of p: the change of p from 0.5 (Fig. 7) to 0.7 (Fig. 9) implies that the diffusion coefficient increases by a factor of $\frac{7}{3}$, as given by Eq. (4.23), while the MFPT decreases by a factor greater than 10. The same effect is observed for the discontinuities (which are therefore not clearly visible in Fig. 9).

V. CONCLUSIONS

We have studied the MFPT problem for stochastic systems driven by a coin-toss square-wave signal. The MFPT for such systems presents the very remarkable feature of being discontinuous and oscillatory. While a nonsmooth behavior of the MFPT has been shown for systems driven by shot noise with constant jumps [19,20], an oscillatory behavior has not been reported in the literature (at least, to our knowledge) for any kind of driving noise. Both features, discontinuities and oscillations, have been found for both free dynamics and dynamics in the presence of an opposing drift (the latter case through computer simulations). For linear driftless dynamics we have been able to obtain exact analytical solutions which, for certain values of the parameters, reduce to the more familiar cases of the ordinary random walk, the persistent random walk, and the continuous-time random walk. This shows the rich structure and generality of the coin-toss square-wave signal.

For dynamics in the presence of a drift (i.e., when there is a potential) we have been able to obtain two analytical approximations valid for certain parameter values. Thus, when the period τ of the random driving signal is small and the probability of persistence p is conveniently rescaled, we arrive at the MFPT for systems driven by Markovian dichotomous noise. On the other hand, when τ is small but the magnitude a of the driving noise is large in such a way that $a^2\tau$ is finite, we obtain another approximation to the problem that we have termed "diffusionlike approximation" because in this case the equation obtained for the MFPT is formally a diffusion equation, though the boundary conditions are modified by the fact that τ is small but nonzero. Moreover when $p = \frac{1}{2}$ and $\tau \equiv 0$ we recover the ordinary diffusion case (i.e., the case of systems driven by Gaussian white noise). In spite of the fact that this is essentially a smoothing approximation that does not take into account discontinuities and oscillations, it gives fairly good results as shown by simulations of the real system. We have also shown an increase in the effective diffusion coefficient as the persistence probability p increases.

We finally mention that these two asymptotic approximations have a parallel in classical probability theory [21] where, depending on parameter values, the binomial distribution can be approximated by either the Poisson distribution (which, in our case, would correspond to a Markovian dichotomous noise) or the Gaussian distribution (which would correspond to the diffusion approximation). From this point of view, our asymptotic approximations can be considered as a proof of the central-limit theorem applied to this problem that shows the so-called *invariance principle* of this theorem, i.e., that the theorem may hold under more and more general assumptions which have not yet been fully explored [22].

ACKNOWLEDGMENTS

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APPENDIX A: SOLUTION OF COUPLED LINEAR DIFFERENCE EQUATION

To find the solution to the problem (3.8) with the boundary conditions (3.3) and (3.9), we begin by defining an operator S such that

$$S^n T^+(x_0) \equiv T^+(x_0 - n) \qquad (n = 0, 1, 2, \cdots). \qquad (A1)$$

Equation (3.8) can then be written as

$$(S-1)^2 T^+(x_0) = -\frac{2q}{p} \qquad (2 \leq x_0 < L), \qquad (A2)$$

and its solution is

$$T^+(x_0) = C + Aj - \frac{q}{p} j^2, \qquad (A3)$$

where

$$j \equiv [L - x_0] \qquad (A4)$$

is the integer part of $L - x_0$ and is thus the number of points in the interval $(x_0, L]$ where the random walker may change its velocity. Depending on the initial position x_0, j can take on the integer values $j = 0, 1, 2, \ldots, N$, where N is the integer part of L, $N \equiv [L]$. In terms of j we can define a "shifted initial position" $\overline{x}_0$,

$$\overline{x}_0 \equiv x_0 + j, \qquad (A5)$$

and from the definition of the operator S we see that

$$T^+(x_0) = S^j T^+(\overline{x}_0). \qquad (A6)$$

We also observe that the condition $L - 1 < x_0 \leq L$ is equivalent to setting $j = 0$. Therefore, in terms of $\overline{x}_0$ the boundary condition (3.3) can be expressed in the form

$$T^+(x_0) = L - \overline{x}_0 \qquad (j=0). \tag{A7}$$

Substituting this condition into Eq. (A3) we find that $C = L - \overline{x}_0$, that is,

$$T^+(x_0) = L - \overline{x}_0 + Aj - \frac{q}{p}j^2 \qquad (j = 0, 1, 2, \ldots, N). \tag{A8}$$

In order to apply the second boundary condition given by Eq. (3.9) we must split the interval $[0,1)$ into two subintervals, $[0, L-N]$ and $(L-N, 1)$. In the first subinterval we have $j = N$, and the substitution of Eq. (A8) (with $j = N$) into Eq. (3.9) yields the result

$$A = \frac{1+Nq}{p} + \frac{q}{1+Nq}[2\overline{x}_0 - L - N] \qquad (N \le \overline{x}_0 \le L). \tag{A9}$$

In the second subinterval we have $j = N-1$, and Eqs. (3.9) and (A8) yield

$$A = \frac{p+Nq}{p} + \frac{q}{p+Nq}[2\overline{x}_0 - L - (N-1)] \qquad (L-1 < \overline{x}_0 < N). \tag{A10}$$

It is not difficult to convince oneself that

$$T^-(L - x_0) = T^+(x_0), \tag{A11}$$

so that one can easily find the expression for T^- from the above expression for T^+.

APPENDIX B: MEAN-FIRST-PASSAGE-TIME DISCONTINUITIES

The discontinuities in the mean-first-passage times noted in Sec. III can easily be evaluated explicitly. At the points

$$x_{0,j} \equiv L - j, \qquad x_{0,j}^+ \equiv \lim_{\epsilon\to 0+} x_{0,j} + \epsilon, \qquad j = 1, \ldots, N \tag{B1}$$

we find

$$T^+(x_{0,j}^+) - T^+(x_{0,j}) = -\frac{\xi}{1+N\xi}\left[N(1+\xi) + N(N-j)\xi + (L-N)\left(1 - \frac{j\xi}{1+(N+1)\xi}\right)\right], \tag{B2}$$

which is negative for all j, and

$$T^-(x_{0,j}^+) - T^-(x_{0,j}) = -\frac{\xi}{1+N\xi}(N-j)\left[N\xi + \frac{\xi}{1+(N+1)\xi}(L-N)\right], \tag{B3}$$

which is also negative for all j. At the points

$$x_{0,k} \equiv k, \qquad x_{0,k}^- \equiv \lim_{\epsilon\to 0-} x_{0,k} + \epsilon, \qquad k = 1, \ldots, N \tag{B4}$$

we find

$$T^+(x_{0,k}) - T^+(x_{0,k}^-) = \frac{\xi}{1+N\xi}(N-k)\left[N\xi + \frac{\xi}{1+(N+1)\xi}(L-N)\right], \tag{B5}$$

which is positive for all k, and

$$T^-(x_{0,k}) - T^-(x_{0,k}^-) = \frac{\xi}{1+N\xi}\left[N(1+\xi) + N(N-k)\xi + (L-N)\left(1 - \frac{k\xi}{1+(N+1)\xi}\right)\right], \tag{B6}$$

which is also positive for all k.

APPENDIX C: SOLUTION OF DIFFUSIONLIKE EQUATIONS WITH ARBITRARY DRIFT

The solution of Eq. (4.22) with (4.26) and (4.27) for arbitrary $f(x)$, z_2, and z_1 is

$$T(x_0) = A + B\int_{x_0}^{z_2} dx\, e^{V(x)/D_r} - \frac{1}{D_r}\int_{x_0}^{z_2} dx\, e^{V(x)/D_r}\int_x^{z_2} dx' e^{-V(x')/D_r}, \tag{C1}$$

where

$$B = \frac{1}{\Delta D_r}\left(\int_{z_1}^{z_2} dx\, e^{V(x)/D_r}\int_x^{z_2} dx' e^{-V(x')/D_r} + \frac{p}{q}a\tau\, e^{V(z_1)/D_r}\int_{z_1}^{z_2} dx\, e^{-V(x)/D_r}\right), \tag{C2}$$

$$A = \frac{p}{q}\, a\tau\, e^{V(z_2)/D_r} B,$$

and

$$\Delta = \int_{z_1}^{z_2} dx\, e^{V(x)/D_r} + \frac{p}{q}\, a\tau\, e^{V(z_2)/D_r} + \frac{p}{q}\, a\tau\, e^{V(z_1)/D_r}. \tag{C3}$$

[1] See, for example, *Noise in Nonlinear Dynamical Systems*, edited by F. Moss and P. V. E. McClintock (Cambridge University Press, Cambridge, England, 1989), Vols. 1, 2, and 3; K. Lindenberg, B. J. West, and G. P. Tsironis, Rev. Solid State Sci. **3**, 143 (1989).
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